On complexity of the quantum Ising model

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Matthew Hastings (Microsoft Research)

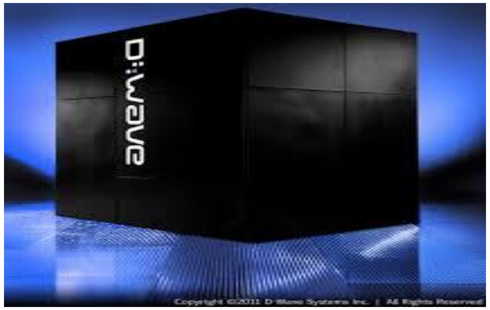
Presenter: David Gosset

Based on
arXiv: 1410.0703
arXiv: 1402.2295

QIP
Sydney
January 16, 2015

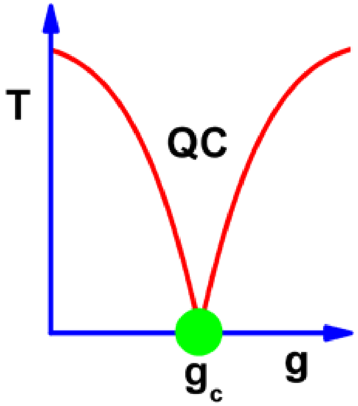
Motivation

Quantum annealing with >100 qubits
Boixo et al, Nature Phys. 10, 218 (2014)

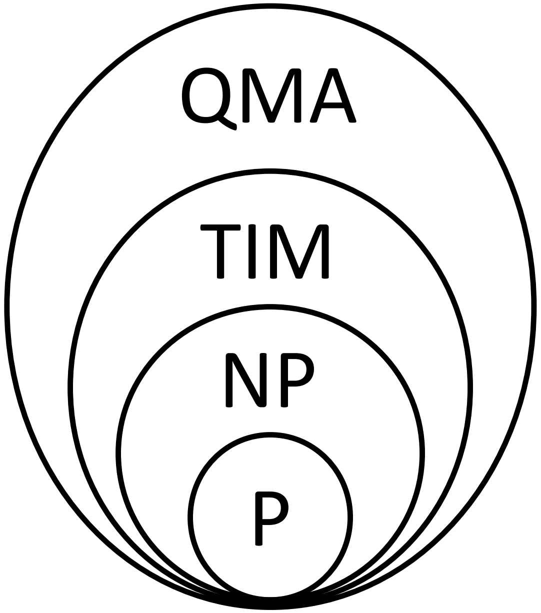


Attempts to solve hard optimization problems such as QUBO

Basic model of phase transitions
Onsager (1944)

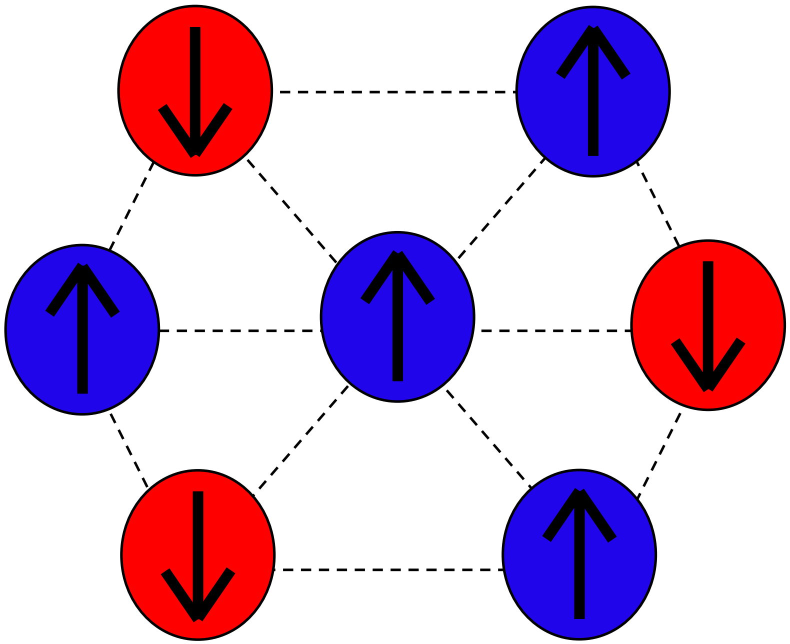


Quantum Hamiltonian Complexity
Cubitt & Montanaro, arxiv:1311.3161



Understand computational hardness of estimating the ground state energy for quantum spin Hamiltonians

Transverse Ising Model (TIM)



- Qubits live at vertices of a graph
- Ising ZZ interactions between nearest neighbor qubits.
- Local magnetic fields along X and Z axes.

$$H = \sum_{u} g_u Z_u + h_u X_u - \sum_{(u,v)} J_{u,v} Z_u Z_v$$

$$Z_u = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad X_u = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Part I

Universality of TIM for quantum annealing

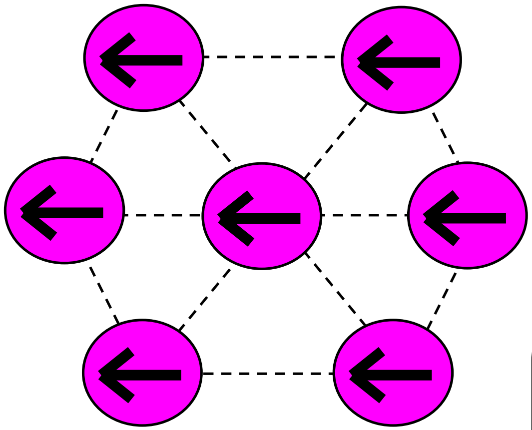
Part II

Computational hardness of estimating the ground state energy of TIM

Part III

Ferromagnetic TIM is easy

Quantum Annealing (Farhi et al 2001)

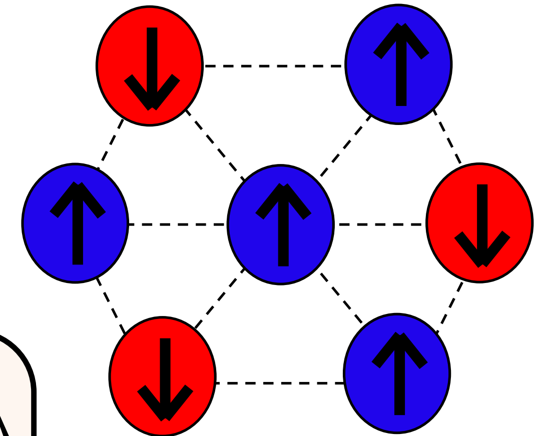


Easy

$$i\frac{\partial|\Psi(t)\rangle}{\partial t} = H(t/T)|\Psi(t)\rangle$$

Unitary evolution

$$0 \le t \le T$$



Hard

Easy: $H(0) = -\sum_u X_u$

Hard: $H(1) = \sum_{(u,v)} J_{u,v}\, Z_u Z_v \; + \sum_u g_u Z_u$

$H(s)$ interpolates between $H(0)$ and $H(1)$

Given an adiabatic path $H(s), 0 \le s \le 1$, how large the evolution time T should be ?

Adiabatic Theorem

$$T \sim \frac{\| \dot{H} \|}{\delta^2} + \frac{\| \dot{H} \|^2}{\delta^3} + \frac{\| \ddot{H} \|}{\delta^2}$$

Here δ is the minimum spectral gap above the ground state of $H(s), 0 \le s \le 1$.

Jansen, Seiler, Ruskai, JMP 48, 102111 (2007)

We need a smooth path with a non-negligible spectral gap

Big open question: what kind of problems can be efficiently solved by the quantum annealing (QA) ?

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Simpler question: can one QA machine efficiently simulate another QA machine ?

simulator QA machine   target QA machine

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TIM Hamiltonians

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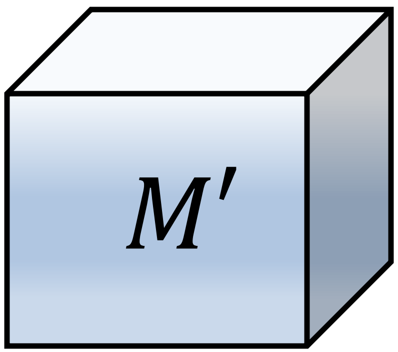

TIM Hamiltonians

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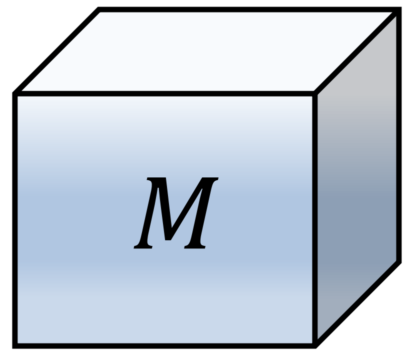


Some fixed target class of Hamiltonians

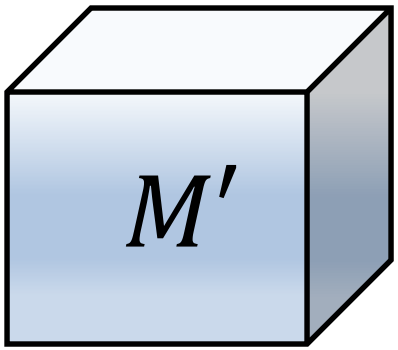
What does efficient simulation mean ?

	Target	Simulator
Adiabatic path	$H(s), 0 \leq s \leq 1$	$H'(s), 0 \leq s \leq 1$
Number of qubits	n	$n' \leq poly(n)$
Minimum spectral gap	δ	$\delta' \geq \delta$
Maximum interaction strength	J	$J' \leq poly(n, \delta^{-1}, J)$
Ground state at $s = 0$	All spins $\vert+\rangle$	All spins $\vert+\rangle$
Ground state at $s = 1$	$\vert\psi\rangle$	$\approx V\vert\psi\rangle$

Here $V: (\mathbf{C}^2)^{\otimes n} \rightarrow (\mathbf{C}^2)^{\otimes n'}$ is a sufficiently simple encoding

When efficient simulation is unlikely:

simulator QA machine



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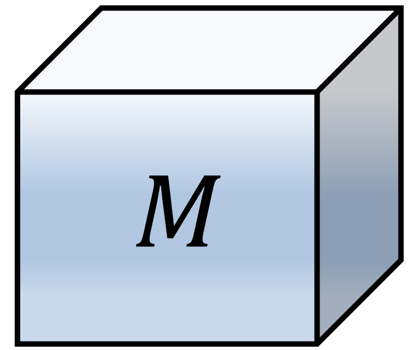

2-local Hamiltonians

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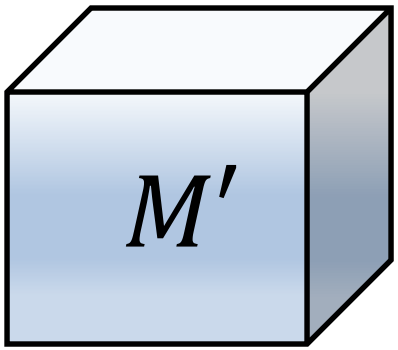

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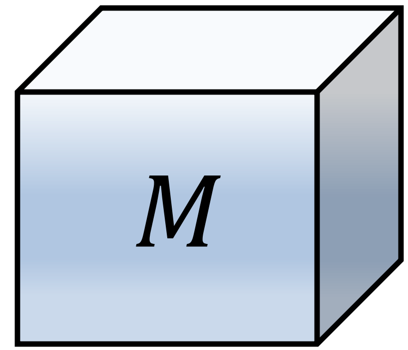

2-local Hamiltonians

⇓

BQP

Aharonov et al (2007)
Oliveira and Terhal (2008)

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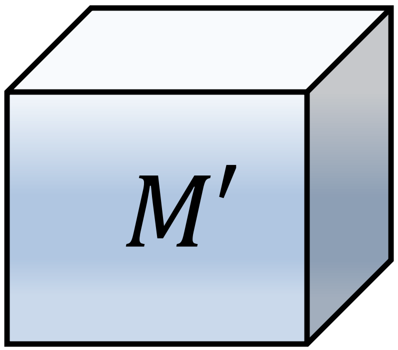

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BQP∩postBPP

SB, DiVincenzo, Oliveira, Terhal (2007)

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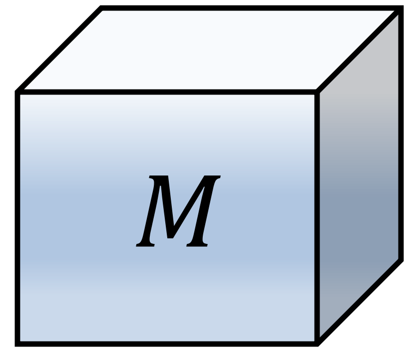

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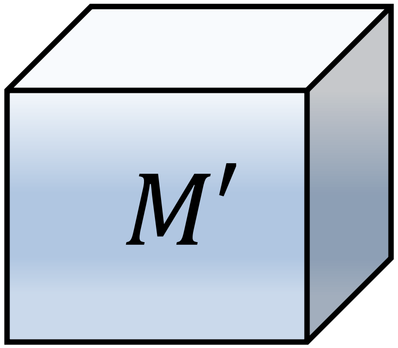

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$\mathrm{BQP} \cap \mathrm{postBPP}$

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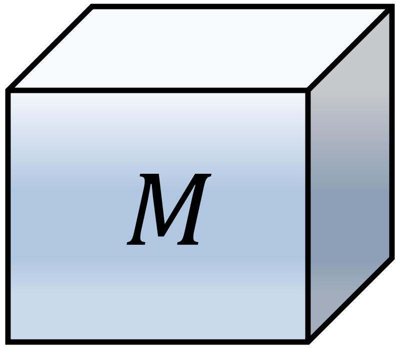
$\neq$

target QA machine



2-local Hamiltonians

⇓

More Powerful

BQP

Aharonov et al (2007)
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Stoquastic k-local Hamiltonians

System of n qubits with a Hamiltonian

$$H = \sum_\alpha H_\alpha$$

Each term H_α acts on at most $k = O(1)$ qubits

1. Matrix elements of H_α in the standard basis are real.

2. Off-diagonal matrix elements of H_α are non-positive:

$$\langle x | H_\alpha | y \rangle \leq 0 \quad \text{for all} \quad x \neq y \in \{0,1\}^k$$

Building blocks for 2-local stoquastic Hamiltonians:

Diagonal : $\pm Z_u, \qquad \pm Z_u Z_v$

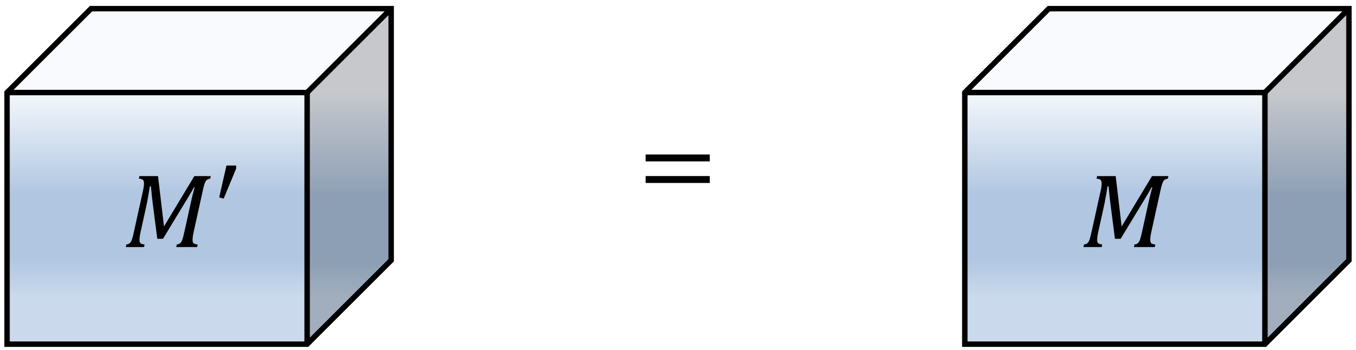
Transverse field: $-X_u$

Elementary interactions: $-X \otimes |0\rangle\langle 0|, \qquad -X \otimes |1\rangle 1|$

$-X \otimes X - Y \otimes Y, \qquad -X \otimes X + Y \otimes Y$

Result 1: universality of TIM for quantum annealing with 2-local stoquastic Hamiltonians

simulator QA machine | target QA machine



TIM Hamiltonians | Stoquastic 2-local Hamiltonians

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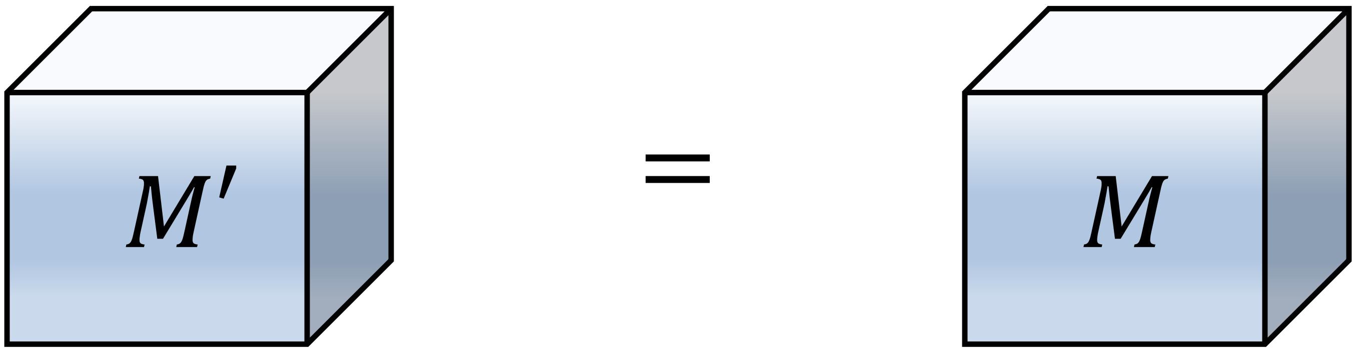

TIM Hamiltonians

Stoquastic 2-local Hamiltonians with k-local diagonal terms

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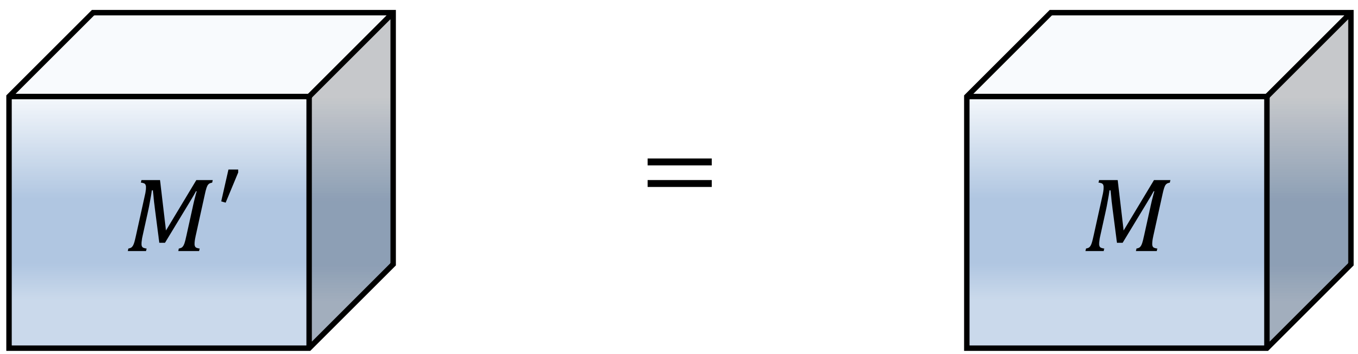


TIM Hamiltonians on degree-3 graphs

Stoquastic 2-local Hamiltonians with k-local diagonal terms

Part II

Computational hardness of estimating the ground state energy of TIM

Ground state energy: $E_0 = \min \langle\psi|H|\psi\rangle$

Local Hamiltonian Problem (LHP):

Input: $(n,\ \ H = \sum_\alpha H_\alpha,\ \ C_{yes} < C_{no})$

Yes-instance: $E_0 \leq C_{yes}$

No-instance: $E_0 \geq C_{no}$

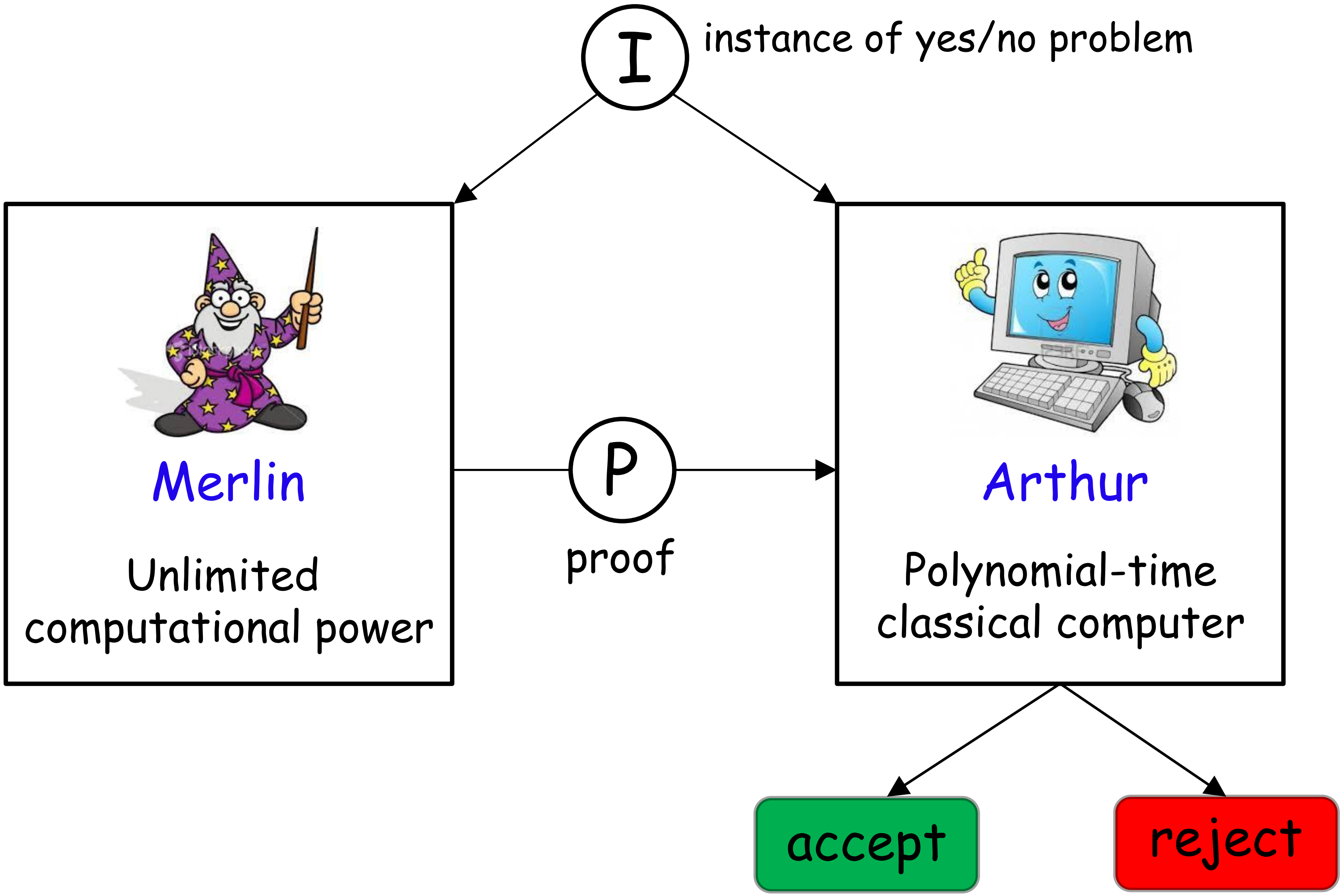
Decide which one is the case.

Promise: $E_0 \notin (C_{yes}, C_{no})$

Normalization: $\|H_\alpha\| \leq poly(n)$, $\quad C_{no} - C_{yes} \geq poly(1/n)$

$\#\text{terms} \leq poly(n)$

Merlin-Arthur games (Babai 1985)



complexity class	A problem belongs to this class if …
NP	yes-instance: Arthur accepts some Merlin's proof no-instance: Arthur rejects any Merlin's proof

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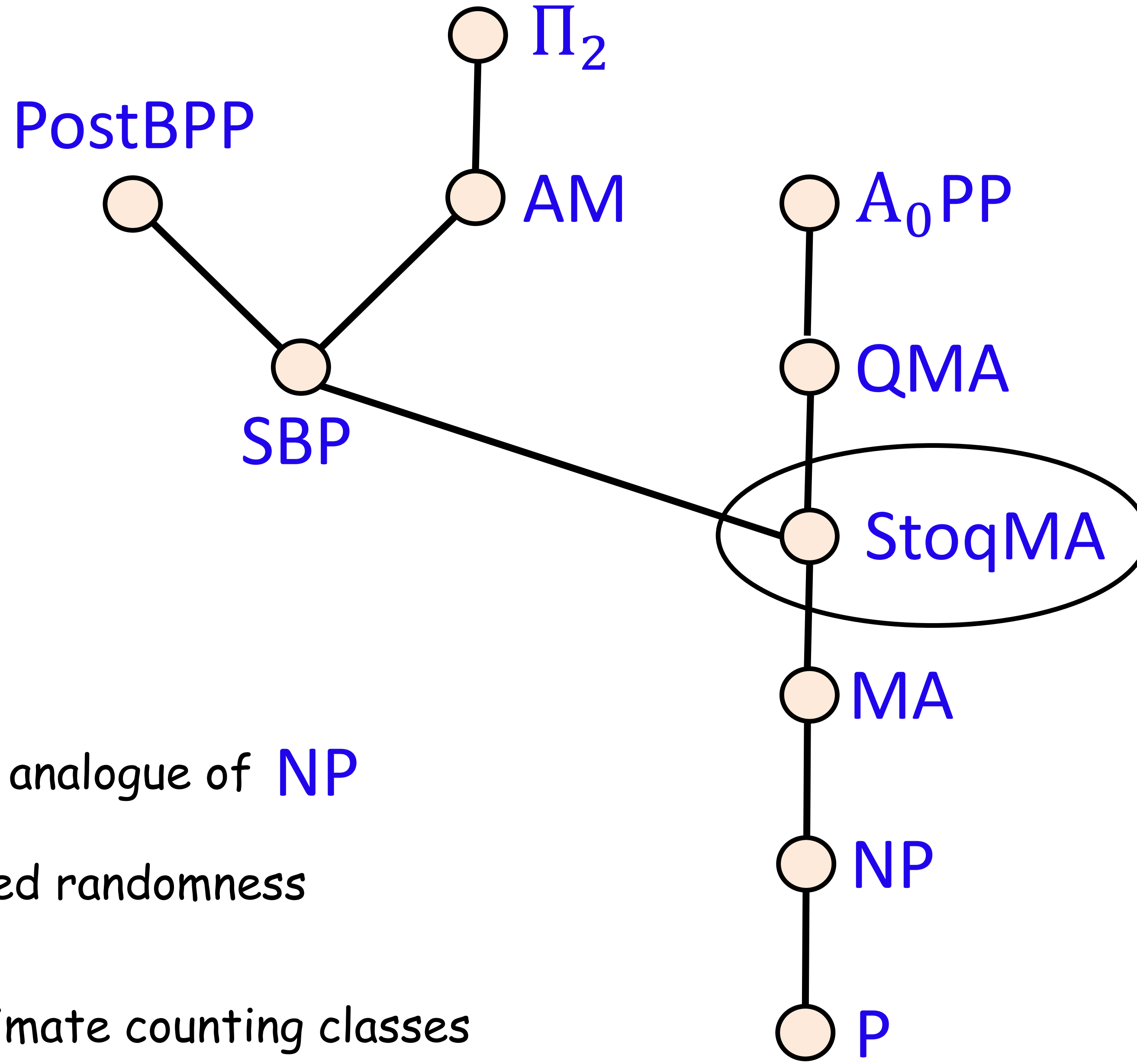
StoqMA	Same as QMA but Arthur can apply only reversible classical gates (CNOT, TOFFOLI) and measure some fixed output qubit in the X-basis. Arthur accepts the proof if the measurement outcome is '+'. Arthur can use $\|0\rangle$ and $\|+\rangle$ ancillas. SB, Bessen, Terhal, arXiv:0611021



MA - randomized analogue of NP

AM=MA + shared randomness

SBP, A_0PP } approximate counting classes

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Result 2: Local Hamiltonian Problem for TIM on degree-3 graphs is StoqMA-complete.

Implications for Cubitt-Montanaro complexity classification of 2-local Hamiltonians (arxiv:1311.3161):

S-LHP: special case of the 2-Local Hamiltonian Problem. All terms in the Hamiltonian must belong to some fixed set S (with arbitrary real coefficients).

Example: $S = \{ Z\otimes Z,\ Z\otimes I,\ X\otimes I\}$ describes TIM-LHP

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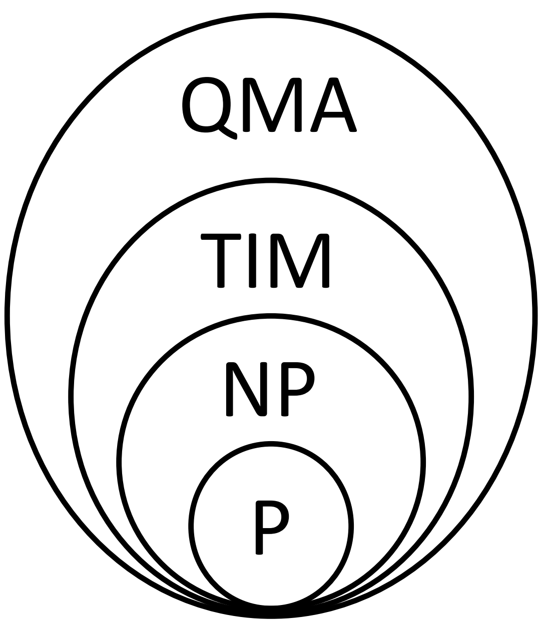

Cubitt-Montanaro (2013):

S-LHP →
- contained in P
- NP-complete
- QMA-complete
- reducible to TIM-LHP

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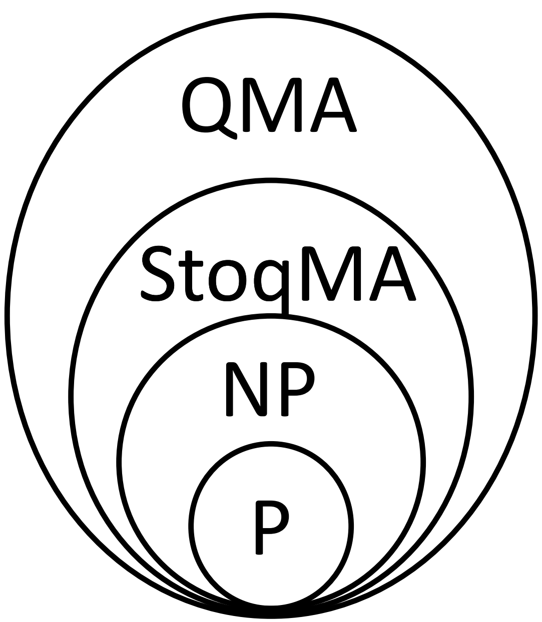

Improved Cubitt-Montanaro:

S-LHP →
- contained in P
- NP-complete
- QMA-complete
- StoqMA-complete

Part III

Ferromagnetic TIM

$$H = \sum_u g\, Z_u + h_u\, X_u - \sum_{(u,v)} J_{u,v}\, Z_u Z_v$$

Uniform Z-field

$J_{u,v} \geq 0$

Classical ferromagnetic Ising model: known results

Computing the minimum energy:

Uniform Z-field: trivial: $\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow$ or $\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow$

Arbitrary Z-fields: $O(n^3)$ algorithm (equivalent to Min Cut problem)

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Computing the partition function $\mathrm{Tr}\, e^{-H}$:

Exact computation is $\#P$-hard, Jerrum & Sinclair (1993)

Uniform Z-field: $O(n^{17}\delta^{-2})$ approximation algorithm
Jerrum & Sinclair (1993)

Arbitrary Z-fields: approximation is $\#BIS$-hard. Unlikely to have poly-time algorithm, Goldberg & Jerrum (2005)

Result 3: Polynomial-time approximation algorithm for the partition function of the ferromagnetic TIM.

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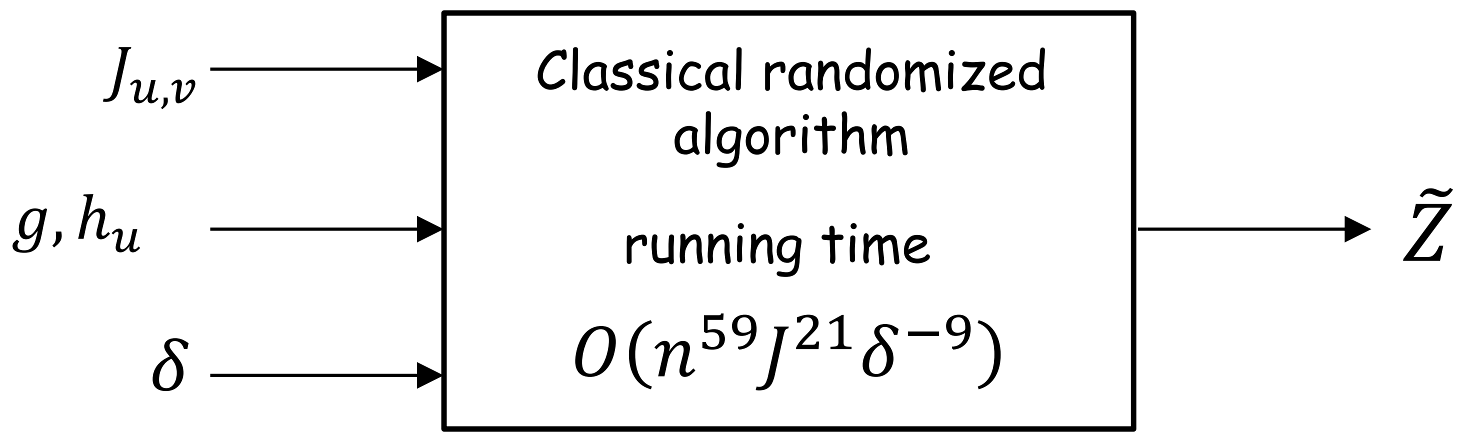

$$(1-\delta)Z \le \tilde{Z} \le (1+\delta)Z \qquad \text{w.h.p.}$$

n = number of spins

$J = \max(J_{u,v}, |h_u|, |g|)$

Result 3: Polynomial-time approximation algorithm for the partition function of the ferromagnetic TIM.

$$Z = \mathrm{Tr}\, e^{-H/T}$$

Implications:

1. The free energy $F(T) = -T\log(Z)$ can be estimated with an additive error δ in time $poly(n, \delta^{-1}, JT^{-1})$

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2. The ground state energy E_0 can be estimated with an additive error δ in time $poly(n, \delta^{-1}, J)$

Sketch of the proofs

Ferromagnetic TIM is easy

$H = -A - B$ $\quad A$ = classical ferromag. Ising model $\quad B$ = transverse field

$$Z = \mathrm{Tr}\, e^{A+B}$$

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$$\underbrace{Z' = \mathrm{Tr}\left(e^{A/r} e^{B/r}\right)^r}_{\text{Trotter-Suzuki approximation to } Z} \qquad r = poly(n)$$

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Trotter-Suzuki approximation to Z

Fact 1: Z' approximates Z with a multiplicative error $O(\delta)$ if

$$r \geq \delta^{-1/2}\left(\|A\|^{3/2} + \|B\|^{3/2}\right)$$

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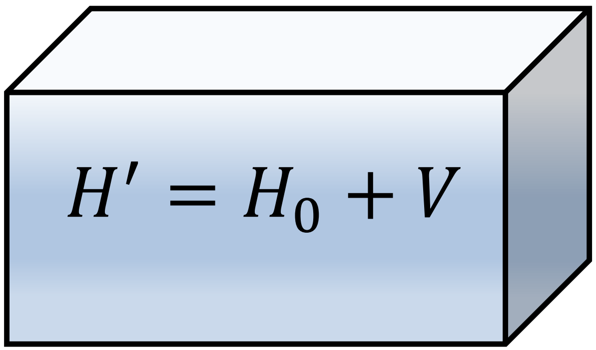
Fact 3: [Jerrum & Sinclair 1993]
The partition function of the classical ferromagnetic Ising model can be approximated in time $O(n^{17}\delta^{-2})$ by a Monte Carlo algorithm.

Sketch of the proofs

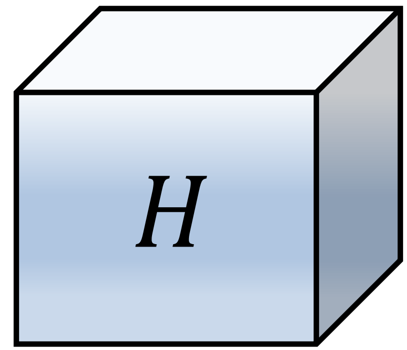
(part I and II)

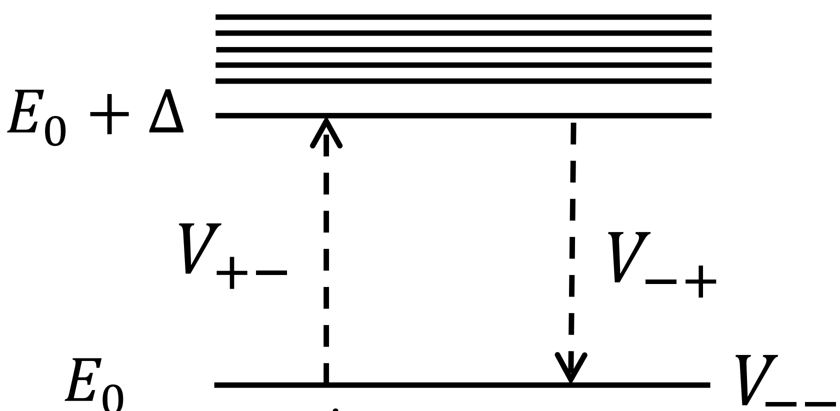
perturbative reductions Kitaev, Kempe, Regev (2004)

simulator Hamiltonian



target Hamiltonian



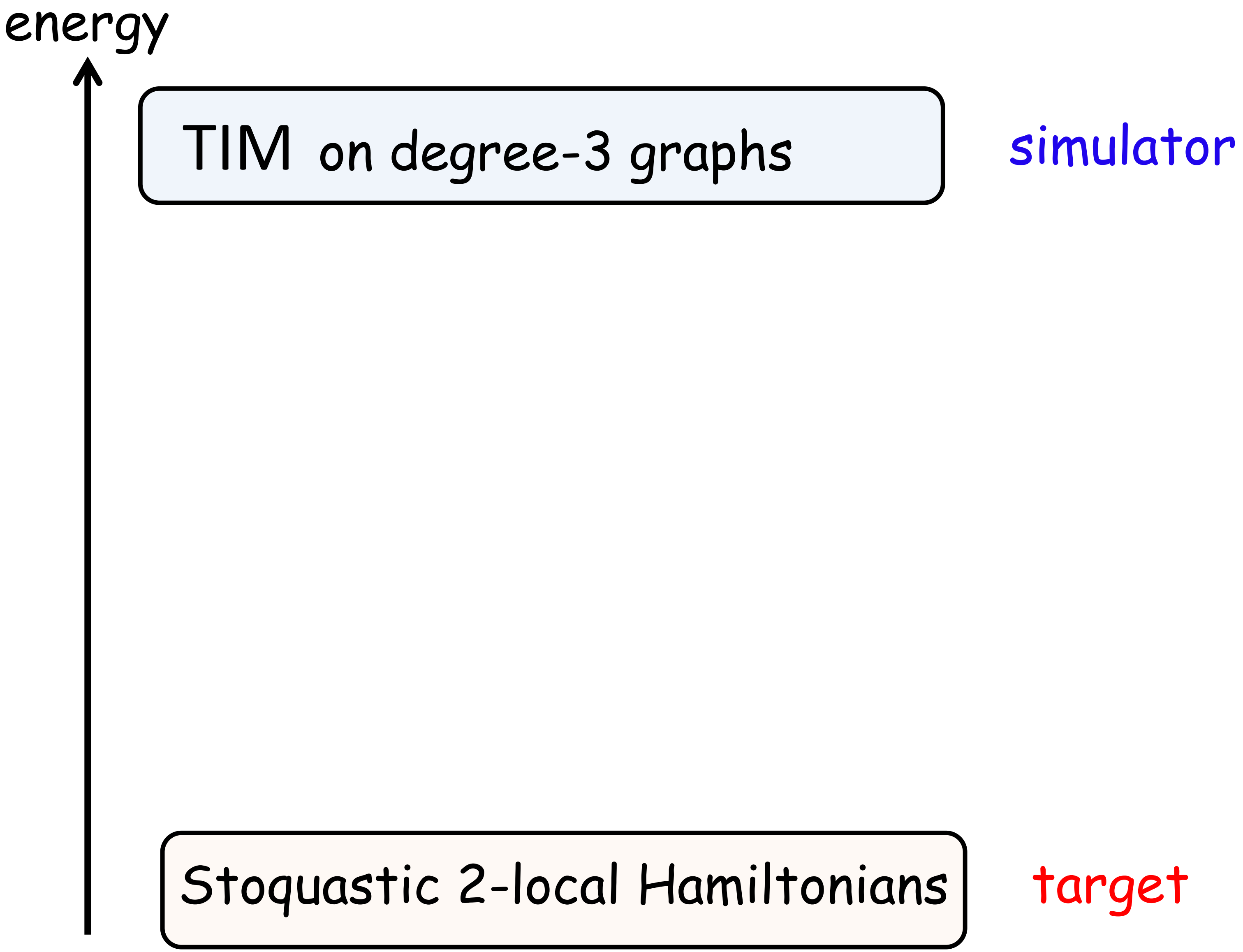


degenerate ground subspace of H_0

$$H \approx H_{\text{eff}} = V_{--} - \Delta^{-1} V_{-+} V_{+-} + \cdots$$

effective low-energy Hamiltonian

Perturbative reductions



Perturbative reductions

energy ↑

TIM on degree-3 graphs — simulator

General TIM

Hard-core dimers

Hard-core bosons (range-2)

Hard-core bosons (range-1)

Hard-core bosons w. controlled hopping

Stoquastic 2-local Hamiltonians — target

Perturbative reductions

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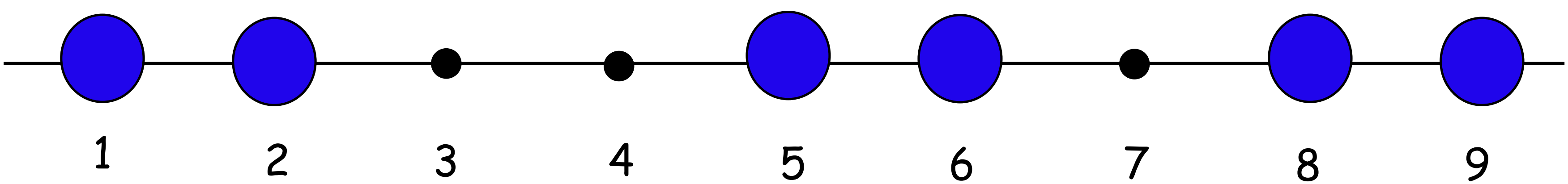
Hard-core dimers model (HCD)

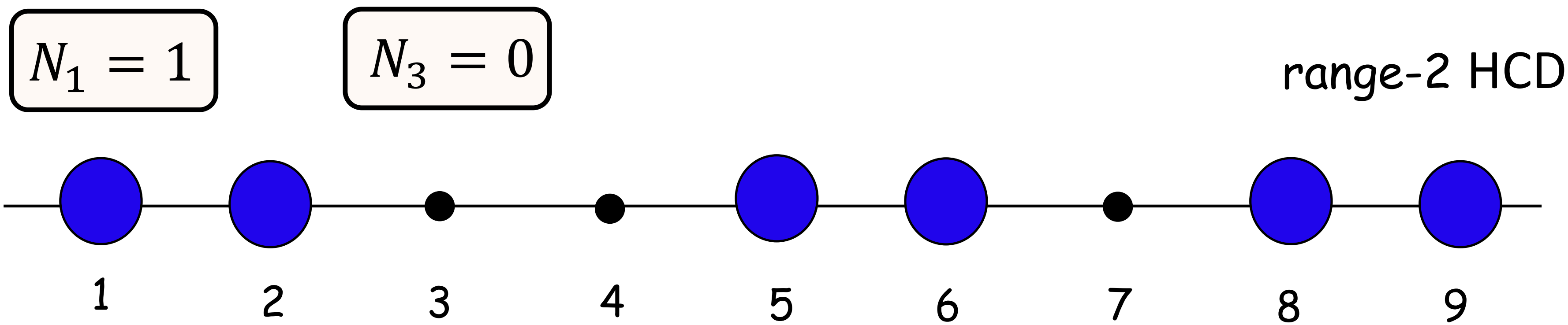
- System of k particles on a fixed graph with n nodes.
- Each site can be either empty or occupied by a particle
- Admissible configurations are nearest-neighbor pairs - dimers
- Dimers must be separated by a fixed distance r – the range

range-2 HCD



Hard-core dimers model (HCD)

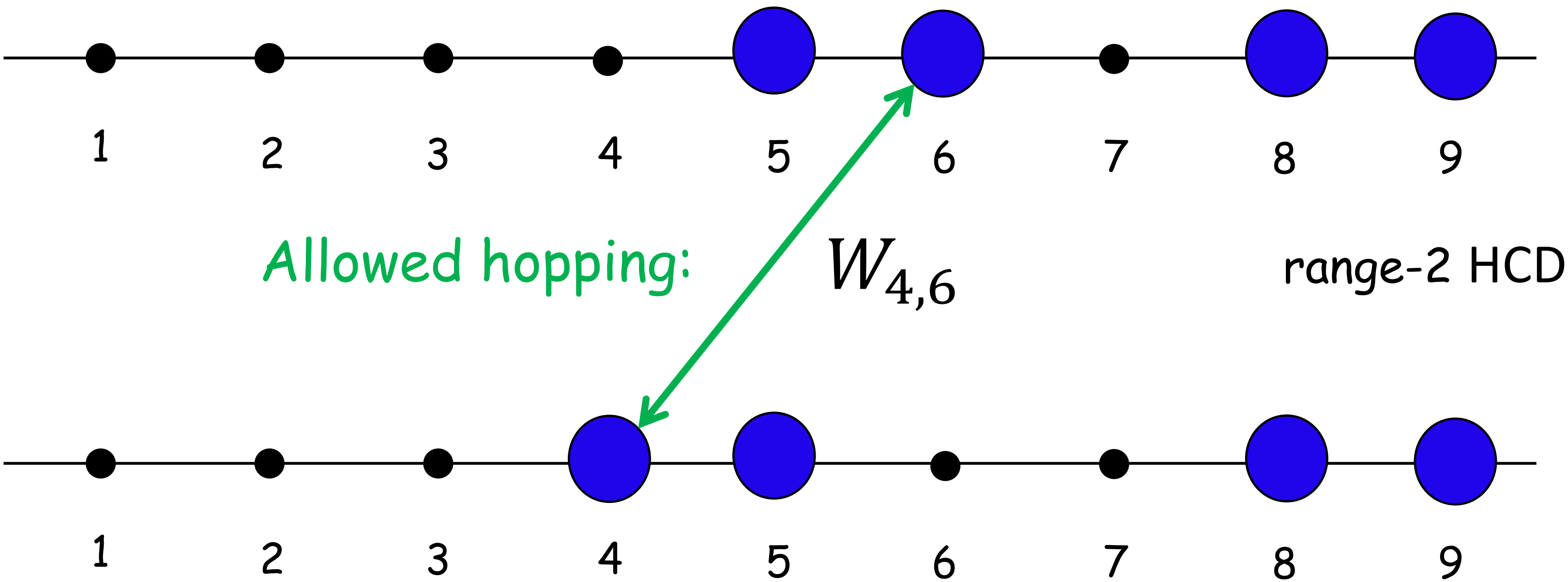
- System of k particles on a fixed graph with n nodes.
- Each site can be either empty or occupied by a particle
- Admissible configurations are nearest-neighbor pairs - dimers
- Dimers must be separated by a fixed distance r – the range



$$H = -t \sum_{u,v} W_{u,v} + \sum_{u} \mu_u N_u + \sum_{u,v} J_{u,v} N_u N_v$$

long-range hopping — on-site chemical potential — two-particle interaction

Dimers can only move locally:



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long-range hopping

on-site chemical potential

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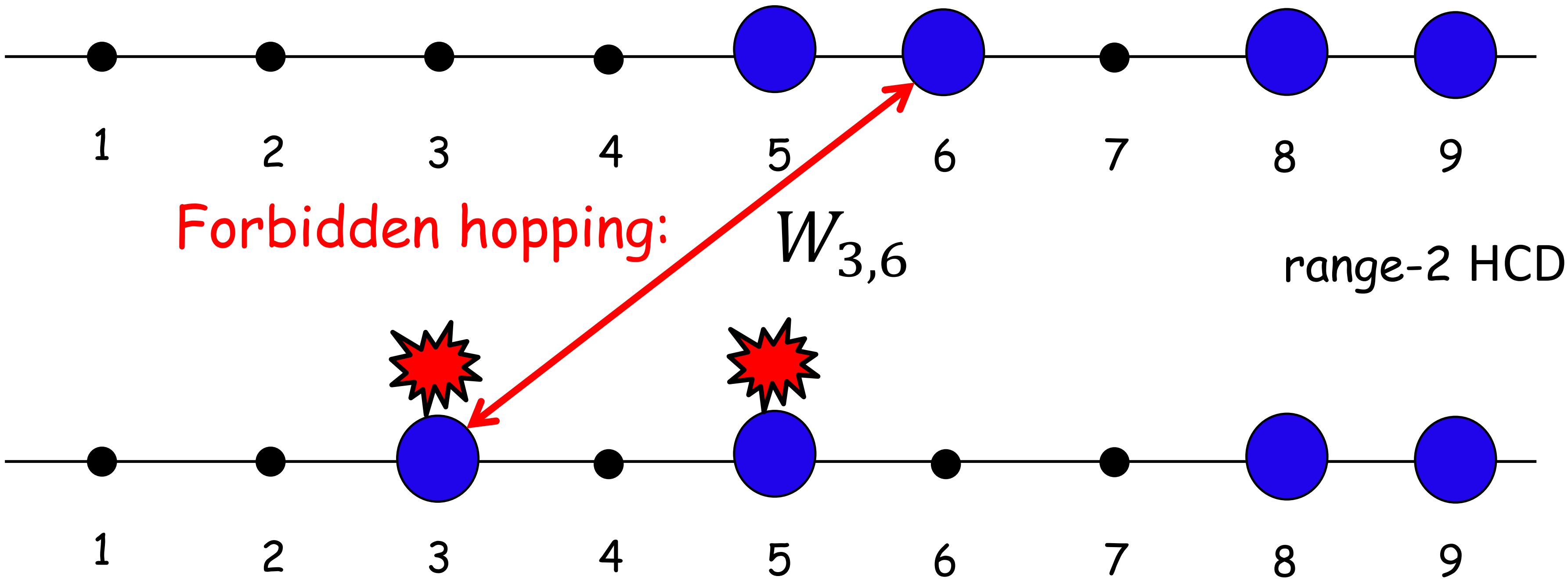

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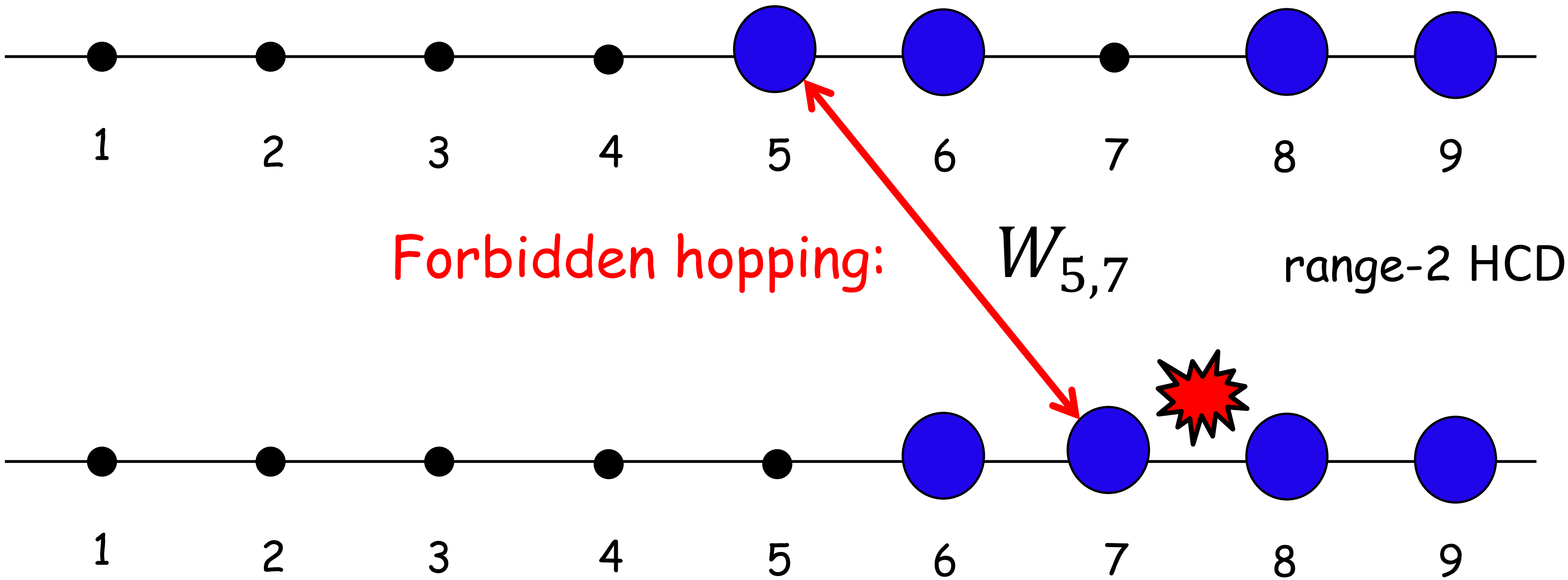
long-range hopping — on-site chemical potential — two-particle interaction

Dimers cannot come too close to each other:



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long-range hopping | on-site chemical potential | two-particle interaction

How the reductions work: overview

energy

TIM on degree-3 graphs

General TIM

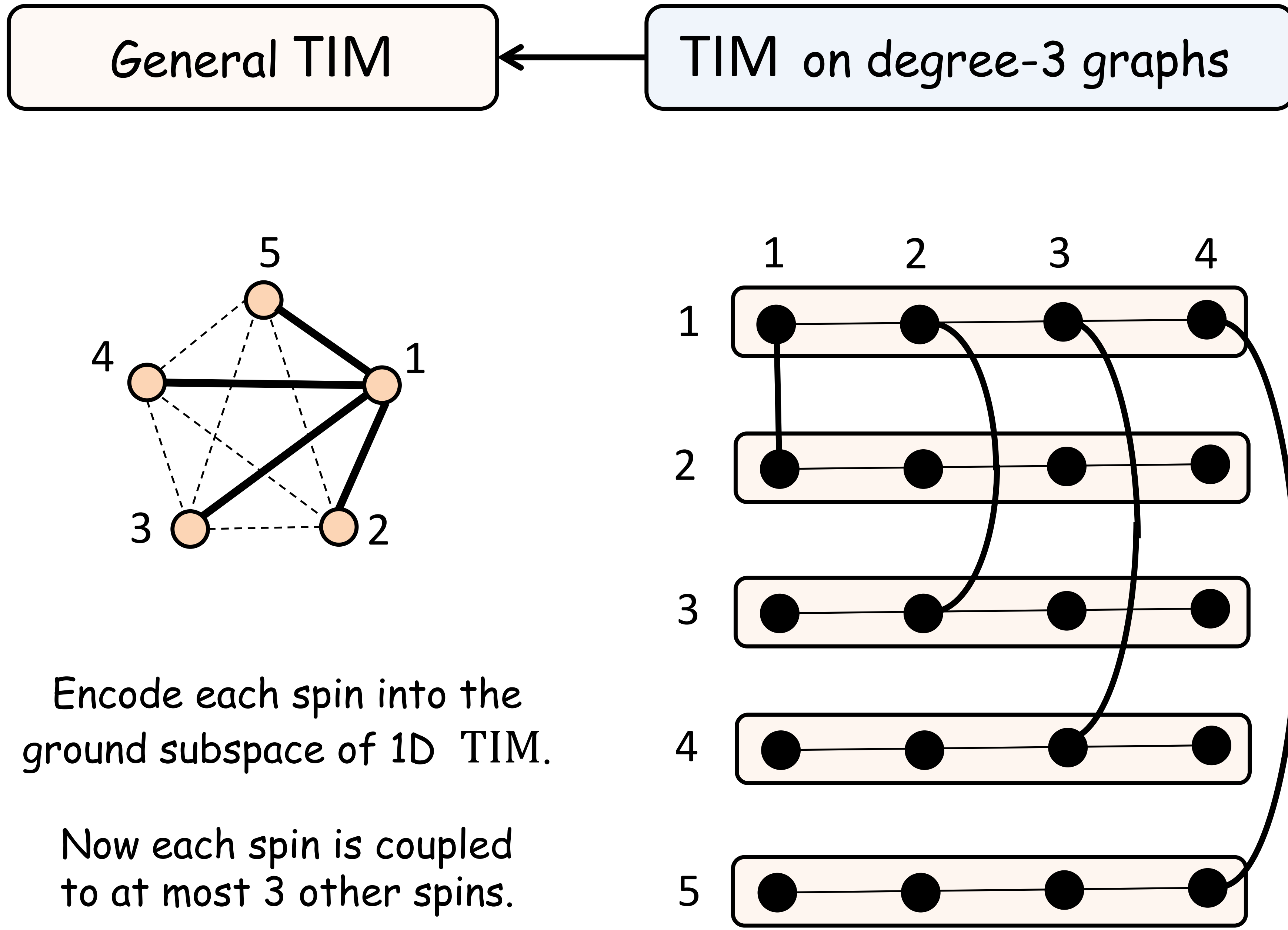
Hard-core dimers (range-3)

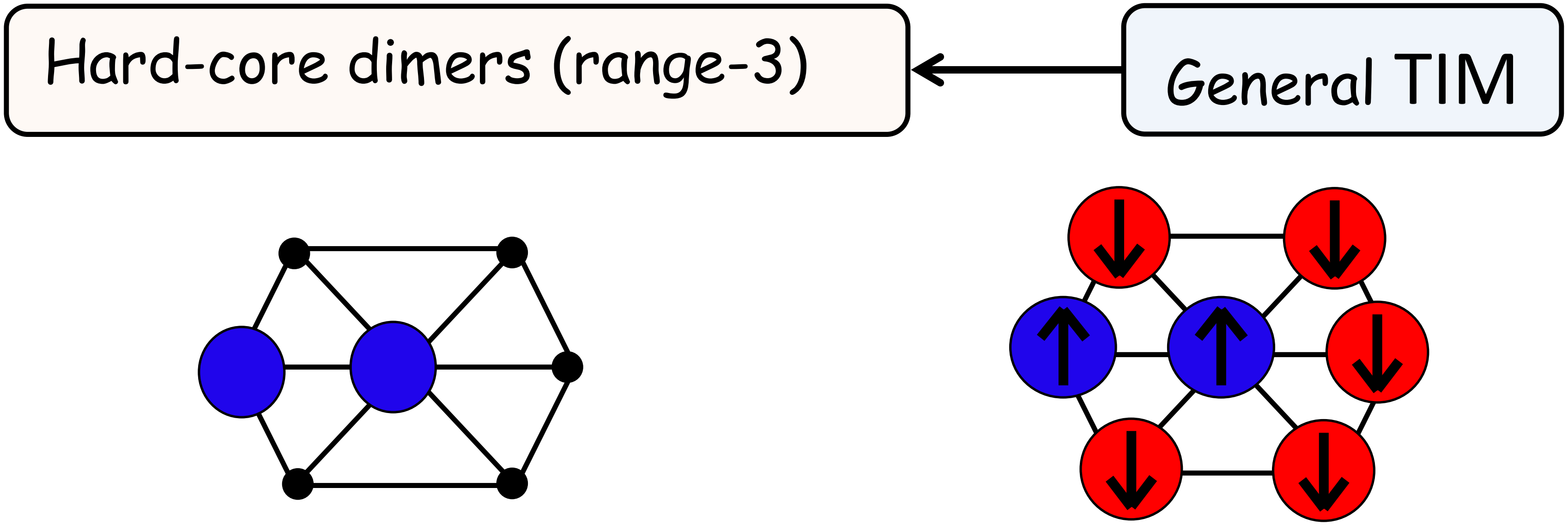
Hard-core bosons (range-2)

Hard-core bosons (range-1)

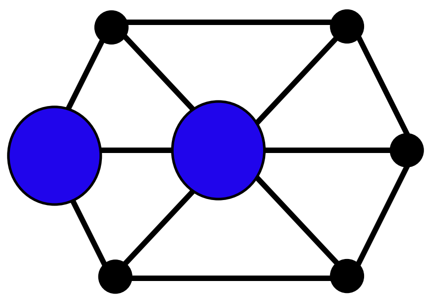
Hard-core bosons w. controlled hopping

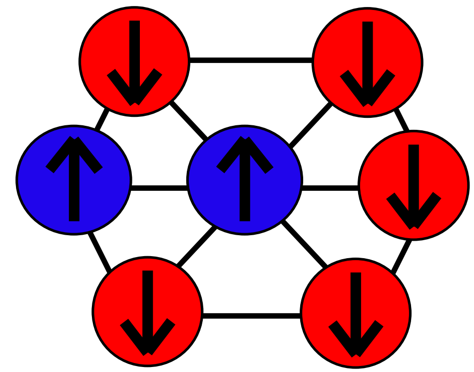
Stoquastic 2-local Hamiltonians


General TIM
TIM on degree-3 graphs
5
4
1
3
2
1 2 3 4
1
2
3
4
5
Encode each spin into the ground subspace of 1D TIM.
Now each spin is coupled to at most 3 other spins.


Hard-core dimers (range-3)
General TIM

Hard-core dimers (range-3) ← General TIM





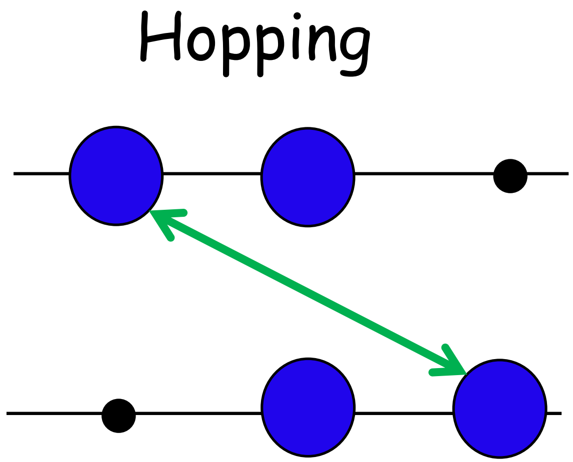
Ising Hamiltonian whose ground states are range-3 dimers:

$$H_0 = \sum_u N_u - 2 \sum_{D(u,v)=1} N_u N_v + \Gamma \sum_{D(u,v)=2} N_u N_v$$

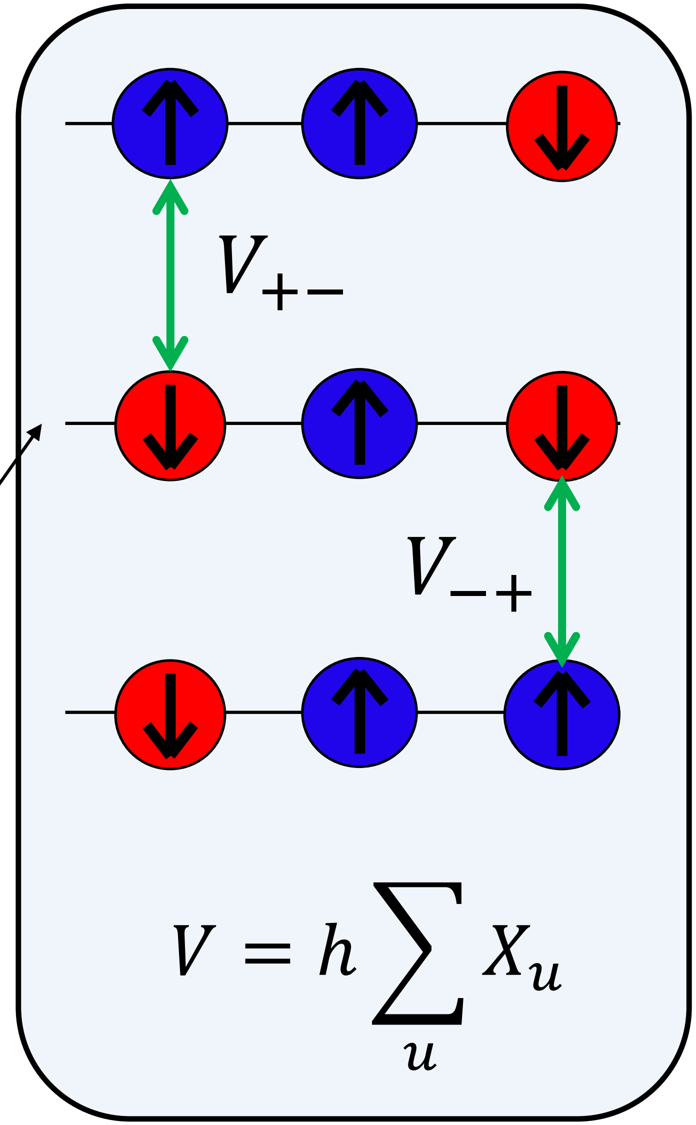
$$N_u = (I + Z_u)/2 \qquad \Gamma = poly(n)$$

$D(u, v)$ - graph distance between sites u, v

Hard-core dimers (range-3) ← General TIM



The intermediate state created by V "remembers" the dimer location. This is why local hopping can emerge from the global transverse field and this is why we need dimers.



Open problems:

Universality of TIM for quantum annealing with k-local stoquastic Hamiltonians for $k > 2$

Is there a subclass of BQP that captures the power of quantum annealing with stoquastic Hamiltonians ?

More efficient algorithms for the ferromagnetic TIM. Can one compute the ground state energy directly without computing the partition function ?

Amplification of the completeness and soundness errors for the class StoqMA